

We give a (remote) quantum gambling scheme that makes use of the fact quantum non-orthogonal states cannot be distinguished with certainty. In the proposed scheme, two participants Alice and Bob can be regarded as playing a game of making guesses on identities of quantum states which are in one of two given non-orthogonal states: if Bob makes a correct (an incorrect) guess on the identity of a quantum state that Alice has sent, he wins (loses). It is shown that the proposed scheme is secure against the non-entanglement-attack. It can also be shown heuristically that the scheme is secure in the case of the entanglement-attack.

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A fundamental property of quantum bits (qubits) that differs from those of classical bits is that unknown qubits can not be copied with unit efficiency [1,2] (the no-cloning theorem). Another related property of qubits is that non-orthogonal qubits can not be distinguished with certainty [3–8]. The no-cloning theorem is the basis for success of Bennett-Brassard 1984 (BB84) quantum key distribution scheme [9]. Therefore, it is interesting to search for quantum protocols utilizing the property that non-orthogonal qubits can not be distinguished with certainty. Bennett’s another quantum key distribution scheme [10] indeed utilizes this property. On the other hand, a (remote) quantum gambling scheme has been found by Goldenberg *et al.*, recently [11].

In this paper we propose another (remote) quantum gambling scheme that makes use of the fact two non-orthogonal qubits can not be distinguished with certainty. In the proposed scheme, two participants Alice and Bob can be regarded as playing a game of making guesses on identities of quantum states which are in one of two given non-orthogonal states; Alice randomly sends one of two non-orthogonal qubits, say,  $|0\rangle$  and  $|\bar{0}\rangle$ . (In this paper,  $|\bar{0}\rangle \equiv (1/\sqrt{2})(|0\rangle + |1\rangle)$  and  $|\bar{1}\rangle \equiv (1/\sqrt{2})(|0\rangle - |1\rangle)$ ). If Bob makes a correct guess, he wins. If not, he loses. Due to the fact that two non-orthogonal qubits can not be distinguished with certainty, it is easy to see that there is no way for Bob to cheat. Alice might try to increase her gain by sending some qubits other than  $|0\rangle$  and  $|\bar{0}\rangle$ . There are two kinds of attacks. In non-entanglement-attacks, qubits sent to Bob are not entangled with Alice’s. In entanglement-attack [12,13] (or EPR attack), qubits sent to Bob are highly entangled with hers. We show the scheme is secure in the case of non-entanglement-attacks. In the case of entanglement-attacks, however, we heuristically show the

security of the scheme. It is true that a quantum cryptographic scheme is of little use without security proof against all ones including entanglement-attacks. And what makes it complicated to prove security of a quantum cryptographic scheme is the entanglement-attack [12,13]. Our security proof of the scheme against the entanglement-attack is heuristic. However, since the idea behind the proof is simple, we believe that a rigorous one will be found, like in the case of the quantum key distributions [14–17].

The difference between our scheme and the original one [11] is that the former relies on that two non-orthogonal states cannot be distinguished with certainty while the latter relies on general quantum mechanical laws. Another difference is that no quantum system needs to be additionally sent in checking steps in our scheme while it needs to be in the original scheme.

Now, let us describe the scheme more precisely.

- (1) Alice randomly chooses one between two non-orthogonal qubits  $|0\rangle$  and  $|\bar{0}\rangle$ , and sends it to Bob [18].
  - (2) On the qubit he receives, Bob performs a measurement by which he can obtain maximal probability  $p$  of correctly guessing the identity of the qubit.
  - (3) On basis of the measurement’s results, he makes a guess on which one the qubit is and announces it to Alice.
  - (4) If he made a correct (an incorrect) guess, Alice announces he has won (lost).
  - (5) When Bob has won, Alice gives him one coin. When he has lost, Bob gives her  $p/(1-p)$  coins.
- However, after the first step, Bob follows the following ones instead of the steps (2)–(5), at randomly chosen instances with a rate  $r$  ( $0 < r \ll 1$ ).
- (These checking steps are similar to those of the Goldenberg *et al.*’s original work [11].)
- (2)’ Bob performs no measurement on the qubit and stores it.
  - (3)’ He announces his randomly chosen guess on identity of the qubit.
  - (4)’ Do the same thing as step (4).
  - (5)’ In the previous step, Alice has actually revealed which one she chose to tell him the qubit is (regardless of her honesty). When it is  $|0\rangle$ , Bob performs  $\hat{S}_z$  ( $\hat{S}_z$  an orthogonal measurement that composes of two projection operators  $|0\rangle\langle 0|$  and  $|1\rangle\langle 1|$  or  $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ ). If outcome is  $|1\rangle$ , Bob announces that he performed  $\hat{S}_z$  and got  $|1\rangle$  as an outcome. Then Alice must give him  $R$  ( $\gg 1$ ) coins. If the outcome is  $|0\rangle$ , Bob says nothing about which measurement he performed and follows step (5). In the case

of  $|\bar{0}\rangle$ , similar things are done with  $\hat{S}_x (= \{|\bar{0}\rangle\langle\bar{0}|, |\bar{1}\rangle\langle\bar{1}|\})$ . In the step (2), it is important for Bob to perform the optimal measurement that assures maximal probability  $p$  of correctly guessing the identity of the qubit in order to assure his maximal gain. Although it is known average information gain is constrained by the Levitin-Holevo's bound [3,4,20], to find the optimal one is not an easy task. Fortunately, however, in the case of two non-orthogonal qubits, the measurement giving maximal information gain is well known [7,8]; a measurement  $\{|\tilde{0}\rangle\langle\tilde{0}|, |\tilde{1}\rangle\langle\tilde{1}|\}$  where  $|\tilde{0}\rangle$  and  $|\tilde{1}\rangle$  are qubits corresponding to a vector  $(1/\sqrt{2})(\hat{z} - \hat{x})$  and  $(1/\sqrt{2})(-\hat{z} + \hat{x})$ , respectively, in the Bloch sphere representation where a single qubit density operator  $\rho_B = (1/2)(\mathbf{1} + \hat{r} \cdot \vec{\sigma})$ . Here  $\mathbf{1}$  is the identity operator,  $\hat{r}$  is a Bloch vector,  $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ , and  $\sigma_x, \sigma_y, \sigma_z$  are the Pauli operators. (see Fig. 1 of Ref. [7] or Fig. 2 of Ref. [8].) Since information gain is maximal if and only if  $p$  is maximal, a measurement with maximal information gain is what maximize  $p$ . Thus the measurement  $\{|\tilde{0}\rangle\langle\tilde{0}|, |\tilde{1}\rangle\langle\tilde{1}|\}$  is the optimal one. For maximal  $p$ , Bob does as the following. When the outcome is  $|\tilde{0}\rangle$  ( $|\tilde{1}\rangle$ ), he makes a guess that the qubit is  $|0\rangle$  ( $|\bar{0}\rangle$ ). Then the probability  $p$  of correctly guessing the qubit is given by  $p = |\langle\tilde{0}|0\rangle|^2 = |\langle\tilde{1}|\bar{1}\rangle|^2 = \cos^2(\pi/8)$ .

Now let us show how each player's average gain is assured. First it is clear by definition that Bob can do nothing better than performing the optimal measurement, as long as Alice prepares the specified qubits. In the scheme, the numbers of coins that Alice and Bob pay are adjusted so that no one gains when Bob's win-probability is  $p$ . Thus Bob's gain  $G_B$  cannot be greater than zero, that is,  $G_B \leq 0$ . Next let us consider Alice's strategy. As noted above, we first show the security against non-entanglement-attacks. In the most general non-entanglement-attacks, Alice randomly generates each qubit in state  $|i\rangle$  with a probability  $p_i$ . Here  $|i\rangle$ s are arbitrarily specified states of qubits,  $i = 1, 2, \dots, N$  and  $\sum_i^N p_i = 1$ . However, since Bob has no knowledge about which  $|i\rangle$  Alice selected at each instance, his treatments on qubits become equal for all qubits. Thus it is sufficient to show the security for a qubit in an arbitrary state  $|j\rangle = a|0\rangle + b|\bar{1}\rangle$ . ( $a$  and  $b$  are some complex numbers with a constraint  $|a|^2 + |b|^2 = 1$ .) First we do it for states within  $z-x$  plane,  $|j\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)|\bar{1}\rangle$ . Later we will generalize the argument to the former case. Let us consider the following. In steps (2)'-(5)', Bob checks at randomly chosen instances that Alice really sent  $|0\rangle$  or  $|\bar{0}\rangle$  by performing measurements  $\hat{S}_x$  or  $\hat{S}_z$ , respectively. If measurement's outcomes are  $|0\rangle$  or  $|\bar{0}\rangle$  ( $|1\rangle$  or  $|\bar{1}\rangle$ ), Alice has passed (not passed) the test. When not passed, Alice must give him  $R(\gg 1)$  coins [19].

Roughly speaking, Alice can do nothing but preparing either  $|0\rangle$  or  $|\bar{0}\rangle$  and honestly tell the identity of the state to him later. Otherwise she sometimes must pay  $R$  coins to him, decreasing her gain. Let us consider this point more precisely. First we estimate upper-bound of Alice's

gain  $G_A$ . It is clear that

$$G_A \leq \max\{G_A^0, G_A^{\bar{0}}\}, \quad (1)$$

where  $\max\{x, y\}$  denotes the maximal one between  $x$  and  $y$ , and  $G_A^0$  ( $G_A^{\bar{0}}$ ) is Alice's gain when she insists the  $|j\rangle$  is  $|0\rangle$  ( $|\bar{0}\rangle$ ). When he has performed the measurements already, he has no way of detecting Alice's cheating. So, Alice's maximal gain in this case is  $p/(1-p)$ . However, it is clear that  $G_A$  is bounded by  $p/(1-p)$  in any case. When he has preserved the qubits following the checking steps, Alice's cheating can be statistically detected. This case contribute to Alice's gain by a largely negative term whose modulus is proportional to product of the rate  $r$  of checking steps, the probability that  $|1\rangle$  or  $|\bar{1}\rangle$  is detected, and the number of coins  $R$  she must pay when it is detected, namely  $-r|\langle 1|j\rangle|^2 R$  or  $-r|\langle \bar{1}|j\rangle|^2 R$ . However, here we should take into account of the fact that Alice obtains partial information about whether Bob has performed the measurement: let  $f_u$  be Alice's estimation of the probability that Bob did not perform the measurement. With no information,  $f_u$  is  $r$ . However, Bob's announced guess gives her partial information on his measurement's result if he did. This information can be used to make better estimate of  $f_u$ . For example, in the case where Alice sends  $|j\rangle$  and Bob performs the optimal measurement  $\{|\tilde{0}\rangle\langle\tilde{0}|, |\tilde{1}\rangle\langle\tilde{1}|\}$ , we obtain using the Bayes's rule that  $f_u = \frac{r}{2}/[\frac{r}{2} + (1-r)|\langle\tilde{0}|j\rangle|^2]$  when his guess is  $|0\rangle$ . However, it is clear that  $f_u \geq r/2$ : when Bob did not perform the measurement, he simply guesses it with equal probabilities regardless of what he received. Thus by the Bayes's rule, Alice can see that there remains a probability greater than  $r/2$  that Bob did not perform the measurement. The relation  $f_u \geq r/2$  also holds for the entanglement-attacks, since it is satisfied for any  $|j\rangle$  as shown above (refer to the related discussion on entanglement-attack later). Combining above facts, we obtain

$$G_A^0 \leq \frac{p}{1-p} - \frac{r}{2}|\langle 1|j\rangle|^2 R, \quad (2)$$

and

$$G_A^{\bar{0}} \leq \frac{p}{1-p} - \frac{r}{2}|\langle \bar{1}|j\rangle|^2 R. \quad (3)$$

By Eqs. (1)-(3), we can see that in order that  $G_A$  is non-negative the following two conditions must be satisfied. (1) Either  $|\langle 1|j\rangle|^2 \sim (1/rR) \ll 1$  (that is,  $|j\rangle \sim |0\rangle$ ) or  $|\langle \bar{1}|j\rangle|^2 \sim (1/rR) \ll 1$  (that is,  $|j\rangle \sim |\bar{0}\rangle$ ). (2) Alice tells him the one between  $|0\rangle$  and  $|\bar{0}\rangle$  which is nearer to  $|j\rangle$ . Otherwise,  $G_A$  will be dominated by the negative second term in the righthand-sides of Eqs. (2) and (3).

Alice might increase her gain by sending a qubit that slightly differs from either  $|0\rangle$  or  $|\bar{0}\rangle$ . However, the gain can be made negligible by making  $R$  large, as we show in the following. Let us consider the case where Alice

prepares  $|j\rangle$  ( $\sim |0\rangle$ ) and later tells him that it is  $|0\rangle$ , for example. In this case,  $G_A$  for a given  $r$  and  $R$  is given by

$$\begin{aligned} G_A &= (1-r)\{|\langle\tilde{0}|j\rangle|^2 \cdot (-1) + |\langle\tilde{1}|j\rangle|^2 \cdot \frac{p}{1-p}\} \\ &\quad - r|\langle 1|j\rangle|^2 R + r|\langle 0|j\rangle|^2 \left\{ \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot \frac{p}{1-p} \right\} \\ &< (1-r)\left\{ \left[\cos\left(\frac{\pi}{8} + \frac{\theta}{2}\right)\right]^2 \cdot (-1) + \left[\sin\left(\frac{\pi}{8} + \frac{\theta}{2}\right)\right]^2 \cdot \frac{p}{1-p} \right\} \\ &\quad - r\left(\sin\frac{\theta}{2}\right)^2 R + 3r. \end{aligned} \quad (4)$$

Here the first (second and third) term in the righthand-sides is due to normal steps (1)-(5) (checking steps (2)'-(5)'). We can check Eq. (4) by verifying that  $G_A < 3r \sim 0$  when  $|j\rangle$  equals  $|0\rangle$ . By the two conditions, we might only consider the case where  $\theta \sim 0$ , and thus we can neglect higher order terms in Eq. (4),

$$G_A < \frac{\alpha}{1-p}\theta - \frac{rR}{4}\theta^2 + 3r, \quad (5)$$

where  $\alpha = \cos(\pi/8)\sin(\pi/8)$  and a small term of order  $\theta^2$  is also neglected. Alice would maximize her gain for given  $r$  and  $R$ . Maximal value of  $G_A$  is obtained when  $\theta = (2\alpha/[1-p])(1/rR)$ .

$$G_A^{max} = \frac{\alpha^2}{(1-p)^2} \frac{1}{rR} + 3r. \quad (6)$$

Bob would minimize  $G_A^{max}$  for a given  $R$ .  $G_A^{max}$  has its minimal value  $2\sqrt{3}\alpha/[(1-p)\sqrt{R}]$  when  $r = \alpha/[(1-p)\sqrt{3R}]$ . Therefore, if Bob chooses  $r = \alpha/[(1-p)\sqrt{3R}]$  then  $G_A$  is bounded by the positive term  $2\sqrt{3}\alpha/[(1-p)\sqrt{R}] \propto 1/\sqrt{R}$  that approaches to zero as  $R$  become large, similarly to the case of the Goldenberg *et al.*'s scheme [11].

Now we argue that using a qubit outside the  $z-x$  plane does not increase Alice's gain: we can see in Eq. (4) that  $G_A$  can only be increased by making the ratio  $|\langle\tilde{1}|j\rangle|^2/|\langle\tilde{0}|j\rangle|^2$  large while keeping  $|\langle 1|j\rangle|^2$  a very small constant. Let us consider some set of  $|j\rangle$ s (not confined in the  $z-x$  plane) that give the same value of  $|\langle 1|j\rangle|^2$ . Bloch vectors of this set make a circle around that of  $|1\rangle$ . We can see by inspection that what gives the maximal value of the ratio  $|\langle\tilde{1}|j\rangle|^2/|\langle\tilde{0}|j\rangle|^2$  lies within the  $z-x$  plane.

Now, let us heuristically argue that the entanglement-attacks [12,13] does not work in the proposed scheme. Let us consider the case where Alice prepares pairs of qubits in an entangled state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|\bar{0}\rangle_B), \quad (7)$$

where  $A$  and  $B$  denote Alice and Bob, respectively. Alice sends qubits with label  $B$  to Bob while storing those

with label  $A$ . If she performs  $\hat{S}_z$ , Bob is given a mixture of  $|0\rangle$  and  $|\bar{0}\rangle$  with equal frequency. Thus if Alice always performs  $\hat{S}_z$ , the attack reduces to a non-entanglement attack where she randomly sends either  $|0\rangle$  or  $|\bar{0}\rangle$ . Let us consider an example illustrating how performing measurements much different from  $\hat{S}_z$  is not of benefit for Alice; we can rewrite Eq. (7) as

$$|\psi\rangle = \frac{\sqrt{2+\sqrt{2}}}{2}|\bar{0}\rangle_A|\alpha\rangle_B + \frac{\sqrt{2-\sqrt{2}}}{2}|\bar{1}\rangle_A|\beta\rangle_B, \quad (8)$$

where  $|\alpha\rangle$  and  $|\beta\rangle$  are normalized ones of  $(|0\rangle + |\bar{0}\rangle)$  and  $(|0\rangle - |\bar{0}\rangle)$ , respectively. Thus if Alice performs  $\hat{S}_x$ , either  $|\alpha\rangle$  or  $|\beta\rangle$  is generated at Bob's site with probabilities given by Eq. (8). However, since all of  $|\langle 1|\alpha\rangle|^2$ ,  $|\langle\bar{1}|\alpha\rangle|^2$ ,  $|\langle 1|\beta\rangle|^2$ , and  $|\langle\bar{1}|\beta\rangle|^2$  are of order of one,  $G_A$  becomes much negative in any case. So Alice would not perform  $\hat{S}_x$ . In fact, if Alice is able to change the qubits between  $|0\rangle$  and  $|\bar{0}\rangle$  as she likes, her cheating will always be successful. However, she is not allowed to do so, since  $|0\rangle\langle 0| \neq |\bar{0}\rangle\langle\bar{0}|$  and Bob's reduced density operator  $\rho_B (= \text{Tr}_A[\rho_{AB}])$  cannot be changed even with entanglement-attacks.

By appropriately choosing her measurement, Alice can generate at Bob's site any  $\{p_i, |i\rangle\langle i|\}$  satisfying  $\sum_i p_i |i\rangle\langle i| = \rho_B$ , where  $\{p_i, |i\rangle\langle i|\}$  denotes a mixture of pure states  $|i\rangle\langle i|$  with relative frequency  $p_i$  (the theorem of Houghston, Jozsa, and Wootters) [21]. Let  $\rho_B = (1/2)(\mathbf{1} + \hat{r} \cdot \vec{\sigma})$ . Since  $\rho_B = \sum_i p_i |i\rangle\langle i|$  and  $|i\rangle\langle i| = (1/2)(\mathbf{1} + \hat{r}_i \cdot \vec{\sigma})$  where  $\hat{r}_i$  is the corresponding Bloch vector, we have  $(1/2)(\mathbf{1} + \hat{r} \cdot \vec{\sigma}) = (1/2)(\mathbf{1} + [\sum_i p_i \hat{r}_i] \cdot \vec{\sigma})$  and thus

$$\hat{r} = \sum_i p_i \hat{r}_i. \quad (9)$$

Therefore, for a given  $\rho_B$  whose Bloch vector is  $\hat{r}$ , Alice can prepare at Bob's site any mixture  $\{p_i, |i\rangle\langle i|\}$  as long as its Bloch vectors  $\hat{r}_i$  satisfy the Eq. (9). However, if Alice always performs a given measurement, the entanglement-attacks reduce to the non-entanglement attacks: outcomes of measurements on entangled pairs do not depend on temporal order of two participant's measurements. So we can confine ourselves to the case where Alice measures first. Then the attack reduces to a non-entanglement attack where Alice generates  $|i\rangle$  with probability  $p_i$ . Alice can only utilize the entanglement by choosing her measurements according to Bob's announced guesses. However, the checking steps also prevent Alice from increasing her gain: she must choose measurement that gives some mixture  $\{p_i, |i\rangle\langle i|\}$  at Bob's site where each  $\hat{r}_i$  is nearly the same as either  $z$  or  $x$ . Otherwise  $G_A$  become dominated by a much negative term involving  $rR$ . Therefore, Alice's freedom in the choice of measurements is negligible and thus she can increase her gain by negligible amounts even with the entanglement-attacks.

Although the proposed scheme can be implemented with currently available technologies, it is very sensitive to errors. So before methods for reducing decoherence, e.g., quantum error correcting codes [22] or decoherence-free subspaces [23] are realized with high performance, the proposed scheme seems to be impractical. And even if such methods are available, errors will remain to be generated with a small rate. Alice might insist that all errors are the residual ones and would not give him the  $R$  coins. Bob's practical solution to this problem is that he aborts the whole protocol if the error rate is greater than the expected residual error rate, as suggested in the original work [11]. Despite these difficulties, however, it is worthwhile to have another application of the fundamental property that non-orthogonal qubits cannot be distinguished with certainty [3–8].

In conclusion, we gave another (remote) quantum gambling scheme that makes use of the fact non-orthogonal states cannot be distinguished with certainty. In the proposed scheme, two participants Alice and Bob can be regarded as playing a game of making guesses on identities of quantum states which are in one of two given non-orthogonal states: if Bob makes a correct (an incorrect) guess on the identity of a quantum state that Alice has sent, he wins (loses). It was shown that the proposed scheme is secure against the non-entanglement-attack. It could also be shown heuristically that the scheme is secure in the case of the entanglement-attack.

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